Twistor approach for harmonic 2-spheres in a loop space

There is a motivation for studying harmonic 2-spheres in a loop space ΩG, where gauge group G is a compact Lie group, namely, there is a conjecture, that a parameter space of based harmonic maps into loop space of a degree k is in a bijective correspondence with a parameter space of k-Yang-Mills connections over S^4 with a group G modulo based gauge transformations. The Atiyah theorem states the bijective correspondence between a parameter space of based holomorphic 2-spheres in a loop space of a degree k and a space of k-instantons over S^4 with a gauge group G modulo based gauge transformations. So, our variant is in some sense realification of this theorem.

Harmonic maps from Riemannian manifold M into Riemannian manifold N are extremal points of a functional $E(\phi) = \int_M |d\phi(p)|^2 vol_g$, where ϕ is varying over all smooth maps between M and N with finite value of $E(\phi)$, where vol_g is a standard volume form on N. In a case of $\mathbb{R}^n$ as a target manifold we have smooth functions, which are in a kernel of a Laplacian operator, as harmonic maps, as usual.

Concerning a loop space of a Lie group G, it is known that a loop space may be isometrically embedded into a Hilbert-Schmidt Grassmannian (infinite dimensional counterpart of Grassmannian manifold, where a vector space is replaced with a Hilbert space), so the task of these harmonic maps' studying comes to harmonic 2-spheres in a Hilbert-Schmidt Grassmannian (a Kähler Hilbert manifold) investigation.

It is known as well, that a flag manifold is a twistor manifold for a Grassmannian manifold (Eels, Salamon). A pseudo-complex manifold $J(N)$ is a twistor manifold for a Riemannian manifold N, if it is smoothly fibred over N and for every Riemann surface M and every pseudo-holomorphic map $\psi : M \to J(N)$ its projection $\phi = \pi \cdot \psi$ to N is a harmonic map.

For $M = \mathbb{C}P^1$ the inverse statement is correct as well, I mean, that for every harmonic map $\phi : \mathbb{C}P^1 \to G_r(\mathbb{C}^n)$ there exists a flag manifold (with some set of indices of intermediate dimensions $\vec{r}$) ans a pseudo-holomorphic map $\psi : \mathbb{C}P^1 \to F_{\vec{r}}(\mathbb{C}^n)$, that $\pi \cdot \psi = \phi$ – initial harmonic map.

This approach allows us to investigate harmonic 2-spheres in a loop space using its embedding into infinite-dimensional Grassmannian and twistor bundle technique for quite explicit understanding how to construct it from pseudo-complex curves in an infinite-dimensional flag manifold.